Plato's Academy

Academy was a suburb of Athens, named after the hero Academos or Ecademos.

“Let no one ignorant of geometry enter here.”

From: Plato, “The Republic”, Book 7 (375 BC)

Undoubtedly; and yet if music and gymnastic are excluded, and the arts are also excluded, what remains?

Well, I said, there may be nothing left of our special subjects; and then we shall have to take something which is not special, but of universal application.

What may that be?

A something which all arts and sciences and intelligences use in common, and which every one first has to learn among the elements of education.

What is that?

The little matter of distinguishing one, two, and three--in a word, number and calculation:--do not all arts and sciences necessarily partake of them?

Yes.

Then the art of war partakes of them?

To the sure.

Then Palamedes, whenever he appears in tragedy, proves Agamemnon ridiculously unfit to be a general. Did you never remark how he declares that he had invented number, and had numbered the ships and set in array the ranks of the army at Troy; which implies that they had never been numbered before, and Agamemnon must be supposed literally to have been incapable of counting his own feet-- how could he if he was ignorant of number? And if that is true, what sort of general must he have been?

I should say a very strange one, if this was as you say.

Can we deny that a warrior should have a knowledge of arithmetic?

Certainly he should, if he is to have the smallest understanding of military tactics, or indeed, I should rather say, if he is to be a man at all.

I should like to know whether you have the same notion which I have of this study?

What is your notion?

It appears to me to be a study of the kind which we are seeking, and which leads naturally to reflection, but never to have been rightly used; for the true use of it is simply to draw the soul towards being.

Will you explain your meaning? he said.

I will try, I said; and I wish you would share the enquiry with me, and say `yes' or `no' when I attempt to distinguish in my own mind what branches of knowledge have this attracting power, in order that we may have clearer proof that arithmetic is, as I suspect, one of them.

Explain, he said.

I mean to say that objects of sense are of two kinds; some of them do not invite thought because the sense is an adequate judge of them; while in the case of other objects sense is so untrustworthy that further enquiry is imperatively demanded.

You are clearly referring, he said, to the manner in which the senses are imposed upon by distance, and by painting in light and shade.

No, I said, that is not at all my meaning.

Then what is your meaning?

When speaking of uninviting objects, I mean those which do not pass from one sensation to the opposite; inviting objects are those which do; in this latter case the sense coming upon the object, whether at a distance or near, gives no more vivid idea of anything in particular than of its opposite. An illustration will make my meaning clearer:-- here are three fingers--a little finger, a second finger, and a middle finger.

Very good.

You may suppose that they are seen quite close: And here comes the point.

What is it?

Each of them equally appears a finger, whether seen in the middle or at the extremity, whether white or black, or thick or thin-- it makes no difference; a finger is a finger all the same. In these cases a man is not compelled to ask of thought the question, what is a finger? for the sight never intimates to the mind that a finger is other than a finger.

True.

And therefore, I said, as we might expect, there is nothing here which invites or excites intelligence.

There is not, he said.

But is this equally true of the greatness and smallness of the fingers? Can sight adequately perceive them? and is no difference made by the circumstance that one of the fingers is in the middle and another at the extremity? And in like manner does the touch adequately perceive the qualities of thickness or thinness, or softness or hardness? And so of the other senses; do they give perfect intimations of such matters? Is not their mode of operation on this wise-- the sense which is concerned with the quality of hardness is necessarily concerned also with the quality of softness, and only intimates to the soul that the same thing is felt to be both hard and soft?

You are quite right, he said.

And must not the soul be perplexed at this intimation which the sense gives of a hard which is also soft? What, again, is the meaning of light and heavy, if that which is light is also heavy, and that which is heavy, light?

Yes, he said, these intimations which the soul receives are very curious and require to be explained.

Yes, I said, and in these perplexities the soul naturally summons to her aid calculation and intelligence, that she may see whether the several objects announced to her are one or two.

True.

And if they turn out to be two, is not each of them one and different?

Certainly.

And if each is one, and both are two, she will conceive the two as in a state of division, for if there were undivided they could only be conceived of as one?

True.

The eye certainly did see both small and great, but only in a confused manner; they were not distinguished.

Yes.

Whereas the thinking mind, intending to light up the chaos, was compelled to reverse the process, and look at small and great as separate and not confused.

Very true.

Was not this the beginning of the enquiry `What is great?' and `What is small?'

Exactly so.

And thus arose the distinction of the visible and the intelligible.

Most true.

This was what I meant when I spoke of impressions which invited the intellect, or the reverse--those which are simultaneous with opposite impressions, invite thought; those which are not simultaneous do not.

I understand, he said, and agree with you.

And to which class do unity and number belong?

I do not know, he replied.

Think a little and you will see that what has preceded will supply the answer; for if simple unity could be adequately perceived by the sight or by any other sense, then, as we were saying in the case of the finger, there would be nothing to attract towards being; but when there is some contradiction always present, and one is the reverse of one and involves the conception of plurality, then thought begins to be aroused within us, and the soul perplexed and wanting to arrive at a decision asks `What is absolute unity?' This is the way in which the study of the one has a power of drawing and converting the mind to the contemplation of true being.

And surely, he said, this occurs notably in the case of one; for we see the same thing to be both one and infinite in multitude?

Yes, I said; and this being true of one must be equally true of all number?

Certainly.

And all arithmetic and calculation have to do with number?

Yes.

And they appear to lead the mind towards truth?

Yes, in a very remarkable manner.

Then this is knowledge of the kind for which we are seeking, having a double use, military and philosophical; for the man of war must learn the art of number or he will not know how to array his troops, and the philosopher also, because he has to rise out of the sea of change and lay hold of true being, and therefore he must be an arithmetician.

That is true.

And our guardian is both warrior and philosopher?

Certainly.

Then this is a kind of knowledge which legislation may fitly prescribe; and we must endeavour to persuade those who are prescribe to be the principal men of our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only; nor again, like merchants or retail-traders, with a view to buying or selling, but for the sake of their military use, and of the soul herself; and because this will be the easiest way for her to pass from becoming to truth and being.

That is excellent, he said.

Yes, I said, and now having spoken of it, I must add how charming the science is! and in how many ways it conduces to our desired end, if pursued in the spirit of a philosopher, and not of a shopkeeper!

How do you mean?

I mean, as I was saying, that arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument. You know how steadily the masters of the art repel and ridicule any one who attempts to divide absolute unity when he is calculating, and if you divide, they multiply, taking care that one shall continue one and not become lost in fractions.

That is very true.

Now, suppose a person were to say to them: O my friends, what are these wonderful numbers about which you are reasoning, in which, as you say, there is a unity such as you demand, and each unit is equal, invariable, indivisible,--what would they answer?

They would answer, as I should conceive, that they were speaking of those numbers which can only be realised in thought.

Then you see that this knowledge may be truly called necessary, necessitating as it clearly does the use of the pure intelligence in the attainment of pure truth?

Yes; that is a marked characteristic of it.

And have you further observed, that those who have a natural talent for calculation are generally quick at every other kind of knowledge; and even the dull if they have had an arithmetical training, although they may derive no other advantage from it, always become much quicker than they would otherwise have been.

Very true, he said.

And indeed, you will not easily find a more difficult study, and not many as difficult.

You will not.

And, for all these reasons, arithmetic is a kind of knowledge in which the best natures should be trained, and which must not be given up.

I agree.

Let this then be made one of our subjects of education. And next, shall we enquire whether the kindred science also concerns us?

You mean geometry?

Exactly so.

Clearly, he said, we are concerned with that part of geometry which relates to war; for in pitching a camp, or taking up a position, or closing or extending the lines of an army, or any other military manoeuvre, whether in actual battle or on a march, it will make all the difference whether a general is or is not a geometrician.

Yes, I said, but for that purpose a very little of either geometry or calculation will be enough; the question relates rather to the greater and more advanced part of geometry--whether that tends in any degree to make more easy the vision of the idea of good; and thither, as I was saying, all things tend which compel the soul to turn her gaze towards that place, where is the full perfection of being, which she ought, by all means, to behold.

True, he said.

Then if geometry compels us to view being, it concerns us; if becoming only, it does not concern us?

Yes, that is what we assert.

Yet anybody who has the least acquaintance with geometry will not deny that such a conception of the science is in flat contradiction to the ordinary language of geometricians.

How so?

They have in view practice only, and are always speaking? in a narrow and ridiculous manner, of squaring and extending and applying and the like-- they confuse the necessities of geometry with those of daily life; whereas knowledge is the real object of the whole science.

Certainly, he said.

Then must not a further admission be made?

What admission?

That the knowledge at which geometry aims is knowledge of the eternal, and not of aught perishing and transient.

That, he replied, may be readily allowed, and is true.

Then, my noble friend, geometry will draw the soul towards truth, and create the spirit of philosophy, and raise up that which is now unhappily allowed to fall down.

Nothing will be more likely to have such an effect.

Then nothing should be more sternly laid down than that the inhabitants of your fair city should by all means learn geometry. Moreover the science has indirect effects, which are not small.

Of what kind? he said.

There are the military advantages of which you spoke, I said; and in all departments of knowledge, as experience proves, any one who has studied geometry is infinitely quicker of apprehension than one who has not.

Yes indeed, he said, there is an infinite difference between them.

Then shall we propose this as a second branch of knowledge which our youth will study?

Let us do so, he replied.

And suppose we make astronomy the third--what do you say?

I am strongly inclined to it, he said; the observation of the seasons and of months and years is as essential to the general as it is to the farmer or sailor.

I am amused, I said, at your fear of the world, which makes you guard against the appearance of insisting upon useless studies; and I quite admit the difficulty of believing that in every man there is an eye of the soul which, when by other pursuits lost and dimmed, is by these purified and re-illumined; and is more precious far than ten thousand bodily eyes, for by it alone is truth seen. Now there are two classes of persons: one class of those who will agree with you and will take your words as a revelation; another class to whom they will be utterly unmeaning, and who will naturally deem them to be idle tales, for they see no sort of profit which is to be obtained from them. And therefore you had better decide at once with which of the two you are proposing to argue. You will very likely say with neither, and that your chief aim in carrying on the argument is your own improvement; at the same time you do not grudge to others any benefit which they may receive.

I think that I should prefer to carry on the argument mainly on my own behalf.

Then take a step backward, for we have gone wrong in the order of the sciences.

What was the mistake? he said.

After plane geometry, I said, we proceeded at once to solids in revolution, instead of taking solids in themselves; whereas after the second dimension the third, which is concerned with cubes and dimensions of depth, ought to have followed.

That is true, Socrates; but so little seems to be known as yet about these subjects.

Why, yes, I said, and for two reasons:--in the first place, no government patronises them; this leads to a want of energy in the pursuit of them, and they are difficult; in the second place, students cannot learn them unless they have a director. But then a director can hardly be found, and even if he could, as matters now stand, the students, who are very conceited, would not attend to him. That, however, would be otherwise if the whole State became the director of these studies and gave honour to them; then disciples would want to come, and there would be continuous and earnest search, and discoveries would be made; since even now, disregarded as they are by the world, and maimed of their fair proportions, and although none of their votaries can tell the use of them, still these studies force their way by their natural charm, and very likely, if they had the help of the State, they would some day emerge into light.

Yes, he said, there is a remarkable charm in them. But I do not clearly understand the change in the order. First you began with a geometry of plane surfaces?

Yes, I said.

And you placed astronomy next, and then you made a step backward?

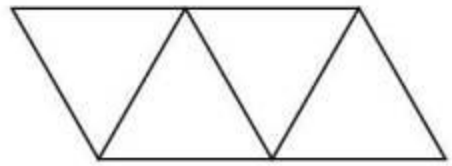
Yes, and I have delayed you by my hurry; the ludicrous state of solid geometry, which, in natural order, should have followed, made me pass over this branch and go on to astronomy, or motion of solids.

Plato's Five Perfect Solids

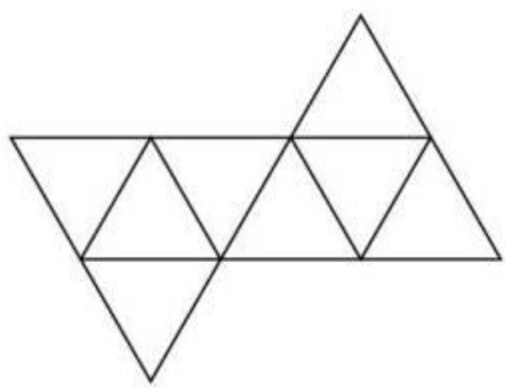
Plato was a Greek philosopher who lived from 427 BC to 347 BC. He was keenly interested in solid geometry and its place in the workings of our universe. To Plato, symmetry was a fundamental property of the universe, and geometry was a tool to know symmetry. Plato is noted for his observations of the set of five regular solids that bear his name. They are polyhedra with symmetrical faces, edges and vertices – perfect solids. He spoke of an atomic universe comprised of four elements: fire, air, earth and water. Four of his solids formed these elements, and the fifth formed the universe itself. And so Plato taught that ideal forms spawn everything in the universe. He thought of nature as a complex system based on diverse instantiations of these ideal forms. The more perfect the form, the more closely it approached whatever truth may lie at the heart of the universe. Only five polyhedrons in the observable universe can be perfectly symmetric.

Tetrahedron

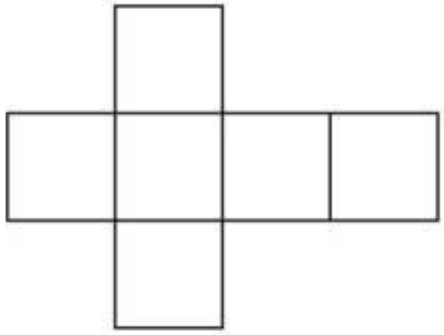
This solid has four triangular faces, four vertices, and six edges. It is dual to itself. The acuteness of its angles led Plato to name it fire.

Octahedron

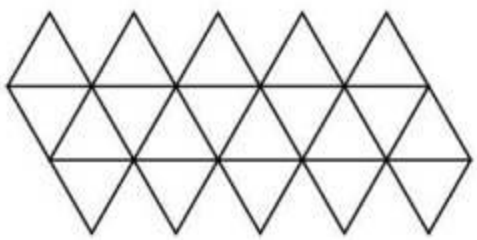
This solid has eight triangular faces, six vertices and twelve edges. It is the dual of the cube. Air is the name given to the octahedron, because it was seen as an intermediate between fire and water.

Cube



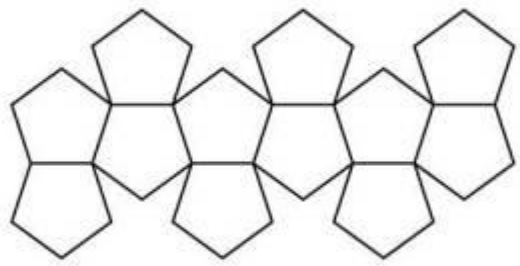
This solid has six square faces, eight vertices and twelve edges. It is the dual of the octahedron. The stability of the cube led Plato to associate it with the element earth.

Icosahedron



This solid has twenty triangular faces, twelve vertices and thirty edges. It is the dual of the dodecahedron. Plato called the icosahedron water.

Dodecahedron

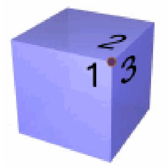
 

This solid has twelve pentagonal faces, twenty vertices and thirty edges. It is the dual of the icosahedron. This is the most mysterious and powerful of the five regular solids. It embodies the other four; Plato therefore said that the dodecahedron is the cosmos. He sensed that it was used by God to embroider the heavens.

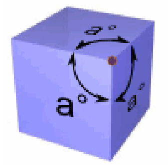
Platonic Solids - Why Five?

Simplest Reason: Angles at a Vertex

The simplest reason there are only 5 Platonic Solids is this:



At each vertex **at least 3 faces** meet (maybe more).

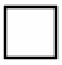


When you add up the internal angles that meet at a vertex, it must be less than 360 degrees (at 360° the shape would flatten out).

We also know that a Platonic Solid's faces are all identical regular polygons:

A regular triangle has internal angles of 60°, so we can have:

- 3 triangles (3×60°=180°)
- 4 triangles (4×60°=240°)
- or 5 triangles (5×60°=300°)



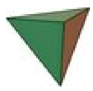
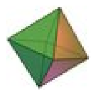
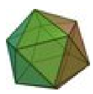
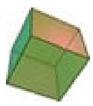
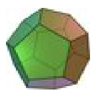
A square has internal angles of 90°, so there is only:

- 3 squares (3×90°=270°)

A regular pentagon has internal angles of 108°, so there is only:

- 3 pentagons (3×108°=324°)

And this is the result:

At each vertex:	Angles at Vertex (Less than 360°)	Solid	
3 triangles meet	180°	tetrahedron	
4 triangles meet	240°	octahedron	
5 triangles meet	300°	icosahedron	
3 squares meet	270°	cube	
3 pentagons meet	324°	dodecahedron	

Anything else has 360° or more at a vertex, which is impossible. Example: 4 regular pentagons (4×108° = 436°), 3 regular hexagons (3×120° = 360°), etc.

אוקלידס---מחבר ה"אלמנטים"

(*) חי באלכסנדריה בסביבות 300 לפנה"ס—מעט ידוע על חייו.

(*) החיבור המונומנטלי שלו—(13 כרכים)

ELEMENTS

היה ספר הלימוד באוניברסיטאות במשך 2000 שנה.

(*) הביטוי הראשון לשיטה האקסיומטית-דדוקטיבית—שפתה של המתימטיקה.

(*) הספר זכה ליותר מאלף הוצאות—שני רק לתנ"ך.

(*) תורגם מיוונית לערבית בזמן הארון אל-רשיד –800 לספירה.

תורגם מערבית ללטינית על ידי נזיר אנגלי ב-1120.

(*) איינשטין אמר כי שתי המתנות היקרות לו בילדותו היו : מצפן ו—הספר של אוקלידס.

Euclid's Elements
Book IX
Proposition 20

Prime numbers are more than any assigned multitude of prime numbers.

Let *A*, *B*, and *C* be the assigned prime numbers.

I say that there are more prime numbers than *A*, *B*, and *C*.

Take the least number *DE* measured by *A*, *B*, and *C*. Add the unit *DF* to *DE*.

Then *EF* is either prime or not.

First, let it be prime. Then the prime numbers *A*, *B*, *C*, and *EF* have been found which are more than *A*, *B*, and *C*.

Next, let *EF* not be prime. Therefore it is measured by some prime number. Let it be measured by the prime number *G*.

I say that *G* is not the same with any of the numbers *A*, *B*, and *C*.

If possible, let it be so.

Now *A*, *B*, and *C* measure *DE*, therefore *G* also measures *DE*. But it also measures *EF*. Therefore *G*, being a number, measures the remainder, the unit *DF*, which is absurd.

Therefore *G* is not the same with any one of the numbers *A*, *B*, and *C*. And by hypothesis it is prime. Therefore the prime numbers *A*, *B*, *C*, and *G* have been found which are more than the assigned multitude of *A*, *B*, and *C*.

Therefore, *prime numbers are more than any assigned multitude of prime numbers.*

משפט: מספר המספרים הראשוניים הוא אינסופי.

תזכורת: מספר ראשוני הוא מספר שאין לו מחלקים (פרט כמובן לעצמו).

הוכחה: נניח בשלילה כי ישנו רק מספר סופי של ראשוניים. נמנה אותם כך:

$$A,\quad B,\quad C,\ldots\ldots,G$$

נכפול את כולם ונוסיף אחד:

$$M = A \times B \times C \times G + 1$$

אזי או ש-:

המספר M הוא מספר ראשוני—אז גמרנו (סתירה!)

או שלמספר M גורם ראשוני, אשר שונה בהכרח מ-

$$A,\quad B,\quad C,\ldots\ldots,G$$

ושוב סתירה!

המספרים המשוכללים

$$6=1+2+3$$

$$28=1+2+4+7+14$$

$$496=1\times2\times2\times2\times2\times31=$$

$$496=1+2+4+8+16+31+62+124+248$$

המספר הבא	**8128**

ארבעת המספרים היו ידועים לאוקלידס והוא פיתח נוסחה למספרים גדולים יותר (ספר 9) אך לא היה יכול לוודא כי מתקיימים התנאים הנחוצים

(זוהי טענה 36 בספר 9—להלן)

List of perfect numbers

From Wikipedia, the free encyclopedia

The following is a list of the known perfect numbers, including the Mersenne prime exponent p which generates them with the expression $2^{p-1} \times (2^p - 1)$ where $2^p - 1$ is a Mersenne prime. All even perfect numbers are of this form. It is not known whether there are any odd perfect numbers.[1] As of 2013 there are 48 known perfect numbers in total.[2][3][4]

Rank	*p*	Perfect number	Digits	Year	Discoverer
1	2	6	1	Known to the Greeks[5]	
2	3	28	2	Known to the Greeks	
3	5	496	3	Known to the Greeks	
4	7	8128	4	Known to the Greeks	
5	13	33550336	8	1456	First seen in a medieval manuscript, *Codex Lat. Monac.*[6]
6	17	8589869056	10	1588	Cataldi[1]
7	19	137438691328	12	1588	Cataldi[1]
8	31	2305843008139952128	19	1772	Euler
9	61	265845599… 953842176	37	1883	Pervushin
10	89	191561942… 548169216	54	1911	Powers
11	107	131640364… 783728128	65	1914	Powers
12	127	144740111… 199152128	77	1876	Lucas
13	521	235627234… 555646976	314	1952	Robinson
14	607	141053783… 537328128	366	1952	Robinson
15	1279	541625262… 984291328	770	1952	Robinson
16	2203	108925835… 453782528	1327	1952	Robinson
17	2281	994970543… 139915776	1373	1952	Robinson
18	3217	335708321… 628525056	1937	1957	Riesel
19	4253	182017490… 133377536	2561	1961	Hurwitz

20	4423	407672717… 912534528	2663	1961	Hurwitz
21	9689	114347317… 429577216	5834	1963	Gillies
22	9941	598885496… 073496576	5985	1963	Gillies
23	11213	395961321… 691086336	6751	1963	Gillies
24	19937	931144559… 271942656	12003	1971	Tuckerman
25	21701	100656497… 141605376	13066	1978	Noll & Nickel
26	23209	811537765… 941666816	13973	1979	Noll
27	44497	365093519… 031827456	26790	1979	Nelson & Slowinski
28	86243	144145836… 360406528	51924	1982	Slowinski
29	110503	136204582… 603862528	66530	1988	Colquitt & Welsh
30	132049	131451295… 774550016	79502	1983	Slowinski
31	216091	278327459… 840880128	130100	1985	Slowinski
32	756839	151616570… 565731328	455663	1992	Slowinski & Gage
33	859433	838488226… 416167936	517430	1994	Slowinski & Gage
34	1257787	849732889… 118704128	757263	1996	Slowinski & Gage
35	1398269	331882354… 723375616	841842	1996	Armengaud, Woltman, et al.
36	2976221	194276425… 174462976	1791864	1997	Spence, Woltman, et al.
37	3021377	811686848… 022457856	1819050	1998	Clarkson, Woltman, Kurowski, et al.
38	6972593	955176030… 123572736	4197919	1999	Hajratwala, Woltman, Kurowski, et al.
39	13466917	427764159… 863021056	8107892	2001	Cameron, Woltman, Kurowski, et al.
40	20996011	793508909… 206896128	12640858	2003	Shafer, Woltman, Kurowski, et al.
41	24036583	448233026… 572950528	14471465	2004	Findley, Woltman, Kurowski, et al.
42	25964951	746209841…	15632458	2005	Nowak, Woltman, Kurowski, et al.

		791088128			
43	30402457	497437765… 164704256	18304103	2005	Cooper, Boone, Woltman, Kurowski, et al.
44	32582657	775946855… 577120256	19616714	2006	Cooper, Boone, Woltman, Kurowski, et al.
45	37156667	204534225… 074480128	22370543	2008	Elvenich, Woltman, Kurowski, et al.
46	42643801	144285057… 377253376	25674127	2009	Strindmo, Woltman, Kurowski, et al.
47	43112609	500767156… 145378816	25956377	2008	Smith, Woltman, Kurowski, et al.
48	57885161	169296395… 270130176	34850340	2013	Cooper, Woltman, Kurowski, et al.

The displayed ranks are among those perfect numbers which are known as of February 2013. Some ranks may change later if smaller perfect numbers are discovered. It is known there is no odd perfect number below 10^{1500}.[7] GIMPS reported that by 20 December 2012 the search for Mersenne primes (and thereby even perfect numbers) became exhaustive up to the 42nd above.[8]

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7. ^ "[1] (http://www.lirmm.fr/~ochem/opn/opn.pdf)" Odd Perfect Numbers Are Greater Than 10^1500. Retrieved 2012-06-02.
8. ^ "GIMPS Milestones Report (http://www.mersenne.org/report_milestones/)". Retrieved 2012-12-21.

External links

- Mersenne Primes: History, Theorems and Lists (http://primes.utm.edu/mersenne/index.html)

Retrieved from "http://en.wikipedia.org/w/index.php?title=List_of_perfect_numbers&oldid=574054036"
Categories: Mathematical tables | Mathematics-related lists

המספרים המשוכללים

Euclid's Elements
Book IX
Proposition 36

If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.

Let as many numbers as we please, *A, B, C,* and *D,* beginning from a unit be set out in double proportion, until the sum of all becomes prime, let *E* equal the sum, and let *E* multiplied by *D* make *FG.*

I say that *FG* is perfect.

For, however many *A*, *B*, *C*, and *D* are in multitude, take so many *E*, *HK*, *L*, and *M* in double proportion beginning from *E*.

Therefore, *ex aequali A* is to *D* as *E* is to *M.* Therefore the product of *E* and *D* equals the product of *A* and *M.* And the product of *E* and *D* is *FG,* therefore the product of *A* and *M* is also *FG.* VII.14 VII.19

Therefore *A* multiplied by *M* makes *FG.* Therefore *M* measures *FG* according to the units in *A.* And *A* is a dyad, therefore *FG* is double of *M.*

But *M*, *L*, *HK*, and *E* are continuously double of each other, therefore *E*, *HK*, *L*, *M*, and *FG* are continuously proportional in double proportion.

Subtract from the second *HK* and the last *FG* the numbers *HN* and *FO,* each equal to the first *E.* Therefore the excess of the second is to the first as the excess of the last is to the sum of those before it. Therefore *NK* is to *E* as *OG* is to the sum of *M, L, KH,* and *E.* IX.35

And *NK* equals *E,* therefore *OG* also equals *M, L, HK, E.* But *FO* also equals *E,* and *E* equals the sum of *A, B, C, D* and the unit. Therefore the whole *FG* equals the sum of *E, HK, L, M, A, B, C, D,* and the unit, and it is measured by them.

I say also that *FG* is not measured by any other number except *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M*, and the unit.

If possible, let some number *P* measure *FG*, and let *P* not be the same with any of the numbers *A*, *B*, *C*, *D*, *E*, *HK*, *L*, or *M*.

And, as many times as *P* measures *FG*, so many units let there be in *Q*, therefore *Q* multiplied by *P* makes *FG*.

But, further, *E* multiplied by *D* makes *FG,* therefore *E* is to *Q* as *P* is to *D.* VII.19

And, since *A, B, C,* and *D* are continuously proportional beginning from a unit, therefore *D* is not measured by any other number except *A, B,* or *C.* IX.13

And, by hypothesis, *P* is not the same with any of the numbers *A, B,* or *C,* therefore *P* does not measure *D.* But *P* is to *D* as *E* is to *Q,* therefore neither does *E*measure *Q.* VII.Def.20

And *E* is prime, and any prime number is prime to any number which it does not measure. Therefore *E* and *Q* are relatively prime. VII.29

But primes are also least, and the least numbers measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent, and *E* is to *Q* as *P* is to *D,* therefore *E* measures *P* the same number of times that *Q* measures *D.* VII.21 VII.20

But *D* is not measured by any other number except *A, B,* or *C,* therefore *Q* is the same with one of the numbers *A, B,* or *C.* Let it be the same with *B.*

And, however many *B*, *C*, and *D* are in multitude, take so many *E*, *HK*, and *L* beginning from *E*.

Now *E, HK,* and *L* are in the same ratio with *B, C,* and *D,* therefore, *ex aequali B* is to *D* as *E* is to *L.* VII.14

Therefore the product of *B* and *L* equals the product of *D* and *E.* But the product of *D* and *E* equals the product of *Q* and *P,* therefore the product of *Q* and *P* also equals the product of *B* and *L.* VII.19

Therefore *Q* is to *B* as *L* is to *P.* And *Q* is the same with *B,* therefore *L* is also the same with *P,* which is impossible, for by hypothesis *P* is not the same with any of the numbers set out. VII.19

Therefore no number measures *FG* except *A, B, C, D, E, HK, L, M,* and the unit.

And *FG* was proved equal to the sum of *A, B, C, D, E, HK, L, M,* and the unit, and a perfect number is that which equals its own parts, therefore *FG* is perfect. VII.Def.22

Therefore, *if as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.*

Q.E.D.

Guide

Summary of the proof

Euclid begins by assuming that the sum of a number of powers of 2 (the sum beginning with 1) is a prime number. Let *p* be the number of powers of 2, and let *s* be their sum which is prime.

$$s = 1 + 2 + 2^2 + \ldots + 2^{p-1}$$

Note that the last power of 2 is 2^{p-1} since the sum starts with 1, which is 2^0.

In Euclid's proof, *A* represents 2, *B* represents 2^2, *C* represents 2^3, and *D* is supposed to be the last power of 2, so it represents 2^{p-1}. Also, *E* represents their sum *s*, and *FG* is the product of *E* and *D*, so it represents $s2^{p-1}$. Let's denote that last by *n*.

$$n = s2^{p-1}$$

The goal is to show that *n* is a perfect number.

In the first part of this proof, Euclid finds some proper divisors of *n* that sum to *n*. These come in two sequences:

$$1, 2, 2^2, \ldots, 2^{p-1}$$

and

$$s, 2s, 2^2s, \ldots, 2^{n-2}s$$

In his proof, the latter are represented by *E*, *HK*, *L*, and finally *M*.

It is clear that each of these is a proper divisor of *n*, and later in the proof Euclid shows that they are the only proper divisors of *n*.

Using the previous proposition, IX.35, Euclid finds the sum of the continued proportion,

$$s + 2s + 2^2 s + \ldots + 2^{n-2} s,$$

to be $2^{n-1}s - s$. But s was the sum $1 + 2 + 2^2 + \ldots + 2^{p-1}$, hence,

$$n = 2^{n-1}s = 1 + 2 + 2^2 + \ldots + 2^{p-1}$$
$$+ s + 2s + 2^2 s + \ldots + 2^{n-2} s$$

Thus, *n* is a sum of these proper divisors.

All that is left to do is to show that they are the only proper divisors of *n*, for then *n* will be the sum of all of its proper divisors, whence a perfect number.

The remainder of the proof is detailed and difficult to follow. It hinges on IX.13 which implies that the only factors of 2^{p-1} are powers of 2, so all the factors of 2^{p-1} have been found. Here's a not-too-faithful version of Euclid's argument. Suppose *n* factors as *ab* where *a* is not a proper divisor of *n* in the list above. In Euclid's proof, *P* represents *a* and*Q* represents *b*.

Since *a* divides $s\,2^{p-1}$, but is not a power of 2, and *s* is prime, therefore *s* divides *a*. Then *b* has to be a power of 2. But then *a* has to be a power of 2 times *s*. But all the powers of 2 times *s* are on the list of known proper divisors. Therefore, the list includes all the proper divisors.

Mersenne primes and perfect numbers

Note that the sum, $s = 1 + 2 + 2^2 + \ldots + 2^{p-1}$, equals $2^p - 1$, by IX.35. As this fact is not needed in the proof, Euclid omits to mention it. Thus, we can restate the proposition as follows:

If $2^p - 1$ is a prime number, then $(2^p - 1)\, 2^{p-1}$ is a perfect number.

Prime numbers of the form $2^p - 1$ have come to be called *Mersenne primes* named in honor of Marin Mersenne (1588-1648), one of many people who have studied these numbers. The four smallest perfect numbers, 6, 28, 496, and 8128, were known to the ancient Greek mathematicians. The Mersenne primes $2^p - 1$ corresponding to these four perfect numbers are 3, 7, 31, and 127, respectively, where the exponents p are 2, 3, 5, and 7, respectively.

The observation that these four exponents are all prime suggests the following two questions:

1. In order for $2^p - 1$ to be prime, is it sufficient for p to be prime?
2. In order for $2^p - 1$ to be prime, is it necessary for p to be prime?

Naturally, the next number to check for primality is $2^{11} - 1$, 2047, which, by a simple search for prime factors is found not to be prime. The number 2047 factors as 23 times 89. Therefore, primality of p is *not* sufficient.

In 1640 Pierre de Fermat (1601-1665) wrote to Mersenne with his investigation of these primes. Fermat found three conditions on p that were necessary for $2^p - 1$ to be prime. One of these conditions answers the second question above— p does have to be prime. Here's a quick argument for that. If p did factor, say as ab, then $2^p - 1$, which is $2^{ab} - 1$, would also factor, namely as

$$2^{ab} - 1 = (2^a - 1)\,(2^{a(b-1)} + 2^{a(b-2)} + \ldots + 2^a).$$

Many mathematicians have studied Mersenne primes since then. A fairly practical testing algorithm was constructed by Lucas in 1876. He showed that the the number $2^p - 1$ is prime if and only if it divides the number $S(p\text{-}1)$, where $S(p\text{-}1)$ is defined recursively: $S(1) = 4$, and $S(n+1) = S(n)^2 - 2$.

The search for more Mersenne primes, and therefore more perfect numbers, continues. It is not known if there are infinitely many or finitely many even perfect numbers. Mersenne primes are scarce, but more continue to be found. There are at least 39 of them, the largest known (as of December 2005) is $2^{30402457} - 1$. It has 9152052 digits. For more information, see the The Great Internet Mersenne Prime Search, GIMPS.

There is a also a question about odd perfect numbers: Are there any? It has been shown that there are no small odd perfect numbers; it is known that odd numbers with fewer then 300 digits are not perfect. It may well be that there are no odd perfect numbers, but to date there is no proof.